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On the mathematical modeling for elastoplastic contact problem and its solution by quadratic programming

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Abstract

In this paper a variational inequality model is created for the elastoplastic contact problem, in which the constraint of the constitutional relation for elastoplastic material and the contact condition are relaxed totally. It gives an effective and strict mathematical modeling for the problem. The quadratic programming is used here for the numerical solution. This algorithm has great advantages of convergence and computational effectiveness over the conventional methods, avoiding the tedious procedure of iterations. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Contact problem; Elastoplastic; Variational inequality model; Quadratic programming

1. Introduction

It is well known that the contact problems widely exist in engineering practice. For example, in most of the mechanical facilities for transmission of motion is carried out through the contact forces; also the large dames are always working under the contact conditions, because of the indispensable construction slits in concrete and the soft layers in the basis. So far, this contact problem has attracted great interest of scientists and many research works on theoretical and numerical analysis can be found in China and abroad (see e.g. Alart and Curnier (1991), Christensen et al. (1998), Michalowski and Mroz (1978), Oden and Carey (1984), Stadter and Weiss (1979), Torstenfelt (1983) and Wang et al. (1982)). However, most of them are too complicated, such as the Newton method (see e.g. Alart and Curnier (1991) and Christensen et al. (1998)), where the conventional step by step loading and progressive iterations are usually adapted.

Generally, the contact surface between the related bodies is indefinite; it depends on the loads. Similar to the elastoplastic problem (see e.g. Chen (1982), Guo and She (1992), Lions (1967) and Zienkiewicz et al. (1975)), the contact problem also belongs to those variational problems with indefinite boundaries. In case of the contact between elastoplastic bodies the constitutive relation and the constraint of contact state are taken as the essential non-linear behaviors which control the entire process of deformation, but in the

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classical variational method (see e.g. Carey and Oden (1984)) the variables are required to be unconstrained in their own regions of definition. As a result, the analysis of these problems is very complicated.

The mathematical modeling of the linear and non-linear elastic contact problems with friction is described by Bjorkman et al. (1995) and Klarbring et al. (1990, 1991) using the variational inequalities, and the conventional methods, such as sequential quadratic programming (see e.g. Bjorkman et al. (1995)) are taken for their numerical solutions.

In this paper, the contact problem and the elastoplastic problem are unified and described by the variational inequality model, from which a functional with the relaxed constitutive relation and the contact restriction is obtained. Furthermore, for the sake of simplicity in computation, the quadratic programming is adopted here to transform the non-linear problem to a complementary linear problem. The presented procedure for numerical solution has great advantages in requirements of CPU time and storage.

2. The differential description for the elastoplastic contact problem

2.1. Contact state equations

In a contact problem (Fig. 1) the law of friction is non-linear. The contact force and the relative displacement may be represented as:

$$P_c = \{P_\xi, P_\zeta, P_n\}^{\mathrm{T}} \tag{1}$$

$$\tilde{\varepsilon} = \begin{Bmatrix} \tilde{\varepsilon}_\xi \\ \tilde{\varepsilon}_\zeta \\ \tilde{\varepsilon}_n \end{Bmatrix} = \begin{Bmatrix} u_\xi^{(1)} - u_\xi^{(2)} \\ u_\zeta^{(1)} - u_\zeta^{(2)} \\ u_n^{(1)} - u_n^{(2)} + a_0 \end{Bmatrix} \tag{2}$$

where a_0 is the clearance between two bodies.

The sliding conditions are:

$$\begin{aligned} \tilde{f}_1 &= |P_\tau| + \overline{\mu} P_n = (P_\xi^2 + P_\zeta^2)^{1/2} + \overline{\mu} P_n \leqslant 0 \\ \tilde{f}_2 &= P_n \leqslant 0 \end{aligned} \tag{3}$$

In the space of the contact forces $P_\xi P_\zeta P_n$, the sliding surface, represented by $\tilde{f}_1 = 0$, is a conical surface (see Fig. 1).

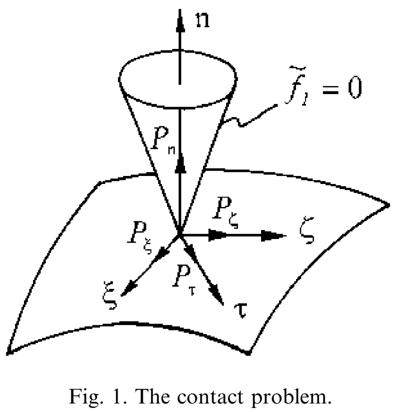

Fig. 1. The contact problem.

Taking the Taylor series expansion for $\tilde{f}_k$ $(k = 1, 2)$ at the point P_c^0, we may obtain

$$\tilde{f}_k = \tilde{f}_k^0 + \left(\frac{\partial \tilde{f}_k}{\partial P_c}\right)^{\mathrm{T}} \mathrm{d}P_c \leqslant 0, \quad k = 1, 2 \tag{4}$$

where $\tilde{f}_k = \tilde{f}_k(P_c^0)$, $\frac{\partial \tilde{f}_k}{\partial P_c} = \frac{\partial \tilde{f}_k}{\partial P_c}\Big|_{P_c = P_c^0}$ are the given values related to the state of the last increment.

Making the partition:

$$\mathrm{d}\tilde{\varepsilon} = \mathrm{d}\tilde{\varepsilon}^{\mathrm{e}} + \mathrm{d}\tilde{\varepsilon}^{\mathrm{s}} \tag{5}$$

where $\mathrm{d}\tilde{\varepsilon}^{\mathrm{e}}$ is the elastic relative displacement, $\mathrm{d}\tilde{\varepsilon}^{\mathrm{s}}$ the relative sliding displacement.

Define the sliding potential functions $\tilde{g}_1$ and $\tilde{g}_2$, corresponding $\tilde{f}_1$, $\tilde{f}_2$ as

$$\tilde{g} = \begin{Bmatrix} \tilde{g}_1 \\ \tilde{g}_2 \end{Bmatrix} = \begin{Bmatrix} (P_\xi^2 + P_\zeta^2)^{1/2} \\ P_n \end{Bmatrix}$$

Then the relative sliding displacement $\mathrm{d}\tilde{\varepsilon}^{\mathrm{s}}$ can be written as

$$\mathrm{d}\tilde{\varepsilon}^{\mathrm{s}} = \sum_{k=1}^{2} \tilde{\lambda}_k \left(\frac{\partial \tilde{g}_k}{\partial P_c}\right) \tag{6}$$

The increments of the contact force and the elastic relative displacement satisfy the Hooke's law

$$\mathrm{d}P_c = \tilde{D}\,\mathrm{d}\tilde{\varepsilon}^{\mathrm{e}} = \tilde{D}\left(\mathrm{d}\tilde{\varepsilon} - \sum_{k=1}^{2} \tilde{\lambda}_k \left(\frac{\partial \tilde{g}_k}{\partial P_c}\right)\right) = \tilde{D}\left(\mathrm{d}\tilde{\varepsilon} - \left(\frac{\partial \tilde{g}}{\partial P_c}\right)^{\mathrm{T}} \tilde{\lambda}\right) \tag{7}$$

where

$$\tilde{D} = \begin{bmatrix} E_\xi & 0 & 0 \\ 0 & E_\zeta & 0 \\ 0 & 0 & E_n \end{bmatrix}$$

E_ξ, E_ζ, E_n are the elastic moduli in the directions ξ, ζ, n.

The contact state equations can be written as

$$\tilde{f}_k = \tilde{f}_k^0 + \left(\frac{\partial \tilde{f}_k}{\partial P_c}\right)^{\mathrm{T}} \tilde{D}\,\mathrm{d}\tilde{\varepsilon} - \sum_{j=1}^{2} \left(\frac{\partial \tilde{f}_k}{\partial P_c}\right)^{\mathrm{T}} \tilde{D}\left(\frac{\partial \tilde{g}_j}{\partial P_c}\right) \tilde{\lambda}_j \leqslant 0 \tag{8}$$

$$\tilde{\lambda}_k \tilde{f}_k = 0, \quad \tilde{\lambda}_k \geqslant 0, \qquad k = 1, 2 \tag{9}$$

2.2. The elastoplastic constitutive equations

It is known that the yield condition can be written as

$$f(\sigma_{ij}, \varepsilon_{ij}^{\mathrm{P}}, k) \leqslant 0 \tag{10}$$

where $f < 0$ for the elastic stage and $f = 0$ for the plastic stage.

According to the flow theory, the increment of the plastic deformation $\mathrm{d}\varepsilon_{ij}^{\mathrm{P}}$ is orthogonal to the surface of the plastic potential $g(\sigma_{ij}, \varepsilon_{ij}^{\mathrm{P}}, k) = 0$ and may be written as $\mathrm{d}\varepsilon_{ij}^{\mathrm{P}} = \lambda \partial g / \partial \sigma_{ij}$, where λ is the parameter of the plastic flow.

For a surface of plastic potential, which can be approximated by several (say m) smooth surfaces, the flow relation is

$$\mathrm{d}\varepsilon_{ij}^{\mathrm{P}} = \sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial g_\alpha}{\partial \sigma_{ij}}$$

where

$$\lambda_\alpha \begin{cases} = 0, & f_\alpha < 0 \\ \geqslant 0, & f_\alpha = 0 \end{cases} \quad \alpha = 1, 2, 3, \ldots, m \tag{11}$$

The increment of deformation consists of two (elastic and plastic) parts as

$$\mathrm{d}\varepsilon_{ij} = \mathrm{d}\varepsilon_{ij}^{\mathrm{e}} + \mathrm{d}\varepsilon_{ij}^{\mathrm{P}} \tag{12}$$

For the elastic part $\mathrm{d}\varepsilon_{ij}^{\mathrm{e}}$ we have

$$\mathrm{d}\sigma_{ij} = D_{ijkl}\,\mathrm{d}\varepsilon_{kl}^{\mathrm{e}} \tag{13}$$

The yield function (10) can be expanded in Taylor series

$$f_\alpha = f_\alpha^0 + \left(\frac{\partial f_\alpha}{\partial \sigma_{ij}}\right)\mathrm{d}\sigma_{ij} + \left(\frac{\partial f_\alpha}{\partial \varepsilon_{ij}^{\mathrm{P}}}\right)\mathrm{d}\varepsilon_{ij}^{\mathrm{P}} + \left(\frac{\partial f_\alpha}{\partial k_i}\right)\mathrm{d}k_i \tag{14}$$

Here $(\partial f_\alpha / \partial \varepsilon_{ij}^{\mathrm{P}})\,\mathrm{d}\varepsilon_{ij}^{\mathrm{P}}$ exists only for strain hardening materials which can be described as

$$\left(\frac{\partial f_\alpha}{\partial \varepsilon_{ij}^{\mathrm{P}}}\right)\mathrm{d}\varepsilon_{ij}^{\mathrm{P}} = -\sum_{\beta=1}^{m} \tilde{t}_{\alpha\beta}\lambda_\beta$$

and $[\tilde{t}_{\alpha\beta}]$ is a positive definite matrix.

When the equal-axial hardening is considered, the hardening function can be linearized as

$$\mathrm{d}k = \tilde{h}\lambda$$

$\tilde{h}$ is a constant.

Then from Eq. (14) we have

$$f_\alpha^0 + \left(\frac{\partial f_\alpha}{\partial \sigma_{ij}}\right)\mathrm{d}\sigma_{ij} - \sum_{\beta=1}^{m} t_{\alpha\beta}\lambda_\beta \leqslant 0 \tag{15}$$

where $t_{\alpha\beta} = \tilde{t}_{\alpha\beta} + \frac{\partial f_\alpha}{\partial k_\beta}\tilde{h}_\beta$.

Substitution of Eqs. (12) and (13) in Eq. (15) yields

$$f_\alpha^0 + \left(\frac{\partial f_\alpha}{\partial \sigma_{ij}}\right) D_{ijkl}\,\mathrm{d}\varepsilon_{kl} - \sum_{\beta}^{m} \left(\frac{\partial f_\alpha}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial g_\beta}{\partial \sigma_{kl}} + t_{\alpha\beta}\right)\lambda_\beta \leqslant 0 \tag{16}$$

From Eqs. (11) and (16) we get

$$\sum_{\beta=1}^{m} \left(\frac{\partial f_\alpha}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial g_\beta}{\partial \sigma_{kl}} + t_{\alpha\beta}\right)\lambda_\beta - \left(\frac{\partial f_\alpha}{\partial \sigma_{ij}}\right) D_{ijkl}\,\mathrm{d}\varepsilon_{kl} - f_\alpha^0 \geqslant 0 \tag{17}$$

$$\lambda_\alpha \left\{ \sum_{\beta=1}^{m} \left(\frac{\partial f_\alpha}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial g_\beta}{\partial \sigma_{kl}} + t_{\alpha\beta}\right)\lambda_\beta - \left(\frac{\partial f_\alpha}{\partial \sigma_{ij}}\right) D_{ijkl}\,\mathrm{d}\varepsilon_{kl} - f_\alpha^0 \right\} = 0 \quad \lambda_\alpha \geqslant 0, \quad \alpha = 1, 2, \ldots, m \tag{18}$$

In the incremental theory, the equilibrium equation, the continuity equation and the boundary conditions are

$$\mathrm{d}\sigma_{ij,j} + \mathrm{d}f_i = 0 \quad \text{in } \Omega \tag{19}$$

$$\mathrm{d}\varepsilon_{ij} = (\mathrm{d}u_{i,j} + \mathrm{d}u_{j,i})/2 \tag{20}$$

$$\mathrm{d}\sigma_{ij} n_j = \mathrm{d}P_i \quad \text{on } \Gamma_P \tag{21}$$

$$\mathrm{d}u_i = \mathrm{d}u_i^0 \quad \text{on } \Gamma_u \tag{22}$$

Eqs. (8), (9) and (17)–(22) are the differential equations for the elastoplastic contact problems.

3. The variational inequality formulation

As for the elastoplastic problems the physical variables, those like the displacements u_i $(i = 1, 2, 3)$, strains ε_{ij}, stresses σ_{ij}, contact displacements $\tilde{\varepsilon}_i$, contact force P_{ci} are the state variables of the system. Some of them can be derived from the basic variables. So the displacements are taken as the state variables and the flow parameter λ, $\tilde{\lambda}$ as the control variables. Defining the following spaces (notation "d" for the increments will be omitted for simplicity):

$$H_1^1(\Omega) = \{u | u \in H_1(\Omega), u|_{\Gamma_u} = u^0\}, \quad H_1^0(\Omega) = \{u | u \in H_1(\Omega), u|_{\Gamma_u} = 0\}$$
$$\overline{H}_1^1(\Omega) = (H_1^1(\Omega))^3, \quad \overline{H}_1^0(\Omega) = (H_1^0(\Omega))^3,$$
$$\overline{L}_2(\Omega) = (L_2(\Omega))^m, \quad \tilde{L}_2(\Omega) = (L_2(\Omega))^2$$
$$\tilde{K} = \{\{u, \lambda, \tilde{\lambda}\} | \{u, \lambda, \tilde{\lambda}\} \in \overline{H}_1^1(\Omega) \times \overline{L}_2(\Omega) \times \tilde{L}_2(\Omega), \quad \lambda_\alpha \geqslant 0, \tilde{\lambda}_\beta \geqslant 0, \alpha = 1, 2, \ldots, m, \beta = 1, 2\}$$

where $H_1(\Omega)$ is the Sobolev space and $L_2(\Omega)$ the Hilbert space.

Consequently, we obtain the following variational inequality formulation equivalent to Eqs. (8), (9) and (17)–(22) as.

Find $\{u, \lambda, \tilde{\lambda}\} \in \tilde{K}$, such that

$$\begin{aligned} &a(u, v-u) - b(v-u, \lambda) + \tilde{a}(u, v-u) - \tilde{b}(v-u, \tilde{\lambda}) + c(r-\lambda, \lambda) - d(u, r-\lambda) + j(r-\lambda) \\ &\quad + \tilde{c}(\tilde{r}-\tilde{\lambda}, \tilde{\lambda}) - \tilde{d}(u, \tilde{r}-\tilde{\lambda}) + \tilde{j}(\tilde{r}-\tilde{\lambda}) \geqslant L(v-u) \quad \forall \{v, r, \tilde{r}\} \in \tilde{K} \end{aligned} \tag{23}$$

where

$$a(u, v) = \int_\Omega \varepsilon_{ij}(u) D_{ijkl} \varepsilon_{kl}(v) \, \mathrm{d}\Omega,$$

$$b(v, r) = \int_\Omega r_\alpha \left(\frac{\partial g_\alpha}{\partial \sigma_{ij}} \right) D_{ijkl} \varepsilon_{kl}(v) \, \mathrm{d}\Omega$$

$$c(\lambda, r) = \int_\Omega \lambda_\alpha \left[\left(\frac{\partial f_\alpha}{\partial \sigma_{ij}} \right) D_{ijkl} \left(\frac{\partial g_\beta}{\partial \sigma_{kl}} \right) + t_{\alpha\beta} \right] r_\beta \, \mathrm{d}\Omega$$

$$d(v, r) = \int_\Omega r_\alpha \left(\frac{\partial f_\alpha}{\partial \sigma_{ij}} \right) D_{ijkl} \varepsilon_{kl}(v) \, \mathrm{d}\Omega$$

$$j(r) = -\int_{\Omega} r_k f_k^0 \,\mathrm{d}\Omega$$

$$L(v) = \int_{\Omega} v^{\mathrm{T}} f \,\mathrm{d}\Omega + \int_{\Gamma_P} v^{\mathrm{T}} P \,\mathrm{d}\Gamma$$

$$\tilde{a}(u, v) = \int_{\Gamma_c} \tilde{\varepsilon}(v) \tilde{D} \tilde{\varepsilon}(u) \,\mathrm{d}\Gamma$$

$$\tilde{b}(u, \tilde{\lambda}) = \int_{\Gamma_c} \sum_k \left(\frac{\partial \tilde{g}_k}{\partial P_c} \right)^{\mathrm{T}} \tilde{D} \tilde{\lambda}_k \tilde{\varepsilon}(v) \,\mathrm{d}\Gamma$$

$$\tilde{c}(\tilde{\lambda}, \tilde{r}) = \int_{\Gamma_c} \sum_{k,j} \tilde{\lambda}_j \left(\frac{\partial \tilde{f}_k}{\partial P_c} \right)^{\mathrm{T}} \tilde{D} \left(\frac{\partial \tilde{g}_j}{\partial P_c} \right) \tilde{r}_k \,\mathrm{d}\Gamma$$

$$\tilde{d}(v, \tilde{r}) = \int_{\Gamma_c} \sum_k \tilde{\lambda}_k \left(\frac{\partial \tilde{f}_k}{\partial P_c} \right)^{\mathrm{T}} \tilde{D} \tilde{\varepsilon}(v) \,\mathrm{d}\Gamma$$

$$\tilde{j}(\tilde{r}) = -\int_{\Gamma_c} \sum_k \tilde{f}_k^0 \tilde{r}_k \,\mathrm{d}\Gamma$$

The equivalence of Eq. (23) and the conventional description of differential equations for the elasto-plastic contact problems can be proved.

Assuming that $\{u, \lambda, \tilde{\lambda}\} \in \tilde{K}$, from Eqs. (19)–(22), we obtain

$$\int_{\Omega} (\sigma_{ij,j} + f_i) \omega_i \,\mathrm{d}\Omega = \int_{\Gamma_P} (\sigma_{ij} n_j - P_i) \omega_i \,\mathrm{d}\Gamma + \sum_{\alpha=1}^{2} \int_{\Gamma_c^{(\alpha)}} \omega_i P_{ci} \,\mathrm{d}\Gamma, \quad \forall \omega \in \overline{H}_1^0(\Omega) \tag{24}$$

Owing to

$$\int_{\Omega} \sigma_{ij,j} \omega_i \,\mathrm{d}\Omega = \int_{\Gamma_P} \sigma_{ij} n_j \omega_i \,\mathrm{d}\Gamma - \int_{\Omega} \sigma_{ij} \varepsilon_{ij} \,\mathrm{d}\Omega, \tag{25}$$

and taking notice of the sign of the local coordinates on the contact boundary, it follows

$$\begin{aligned} \sum_{\alpha=1}^{2} \int_{\Gamma_c^{(\alpha)}} \omega_i P_{ci} \,\mathrm{d}\Gamma &= \int_{\Gamma_c^{(1)}} \omega_i P_{ci} \,\mathrm{d}\Gamma + \int_{\Gamma_c^{(2)}} \omega_i P_{ci} \,\mathrm{d}\Gamma = \int_{\Gamma_c^{(2)}} \omega_i^{(2)} P_{ci} \,\mathrm{d}\Gamma - \int_{\Gamma_c^{(1)}} \omega_i^{(1)} P_{ci} \,\mathrm{d}\Gamma \\ &= \int_{\Gamma_c} (\omega_i^{(2)} - \omega_i^{(1)}) P_{ci} \,\mathrm{d}\Gamma = \int_{\Gamma_c} \tilde{\varepsilon}_i(\omega) P_{ci} \,\mathrm{d}\Gamma \end{aligned} \tag{26}$$

Substitution of Eqs. (25) and (26) in Eq. (24) leads to

$$\int_{\Omega} \sigma_{ij} \varepsilon_{ij}(\omega) \,\mathrm{d}\Omega + \int_{\Gamma_c} \tilde{\varepsilon}_i(\omega) P_{ci} \,\mathrm{d}\Gamma = \int_{\Omega} \omega_i f_i \,\mathrm{d}\Omega + \int_{\Gamma_P} \omega_i P_i \,\mathrm{d}\Gamma \tag{27}$$

or in vector form

$$\int_\Omega \varepsilon^{\mathrm{T}}(\omega) D\left(\varepsilon(u) - \sum_j \left(\frac{\partial g_j}{\partial \sigma}\right)\lambda_j\right) \mathrm{d}\Omega + \int_{\Gamma_c} \tilde{\varepsilon}^{\mathrm{T}}(\omega)\tilde{D}\left[\tilde{\varepsilon}(u) - \sum_j \left(\frac{\partial \tilde{g}_j}{\partial P_c}\right)\tilde{\lambda}_j\right] \mathrm{d}\Gamma$$
$$= \int_\Omega \omega^{\mathrm{T}} f \,\mathrm{d}\Omega + \int_{\Gamma_P} \omega^{\mathrm{T}} P \,\mathrm{d}\Gamma \tag{28}$$

If we set $v = u + \omega \in \overline{H}_1^1(\Omega)$, the above equation will be

$$a(u, v-u) - b(v-u, \lambda) + \tilde{a}(u, v-u) - \tilde{b}(v-u, \tilde{\lambda}) = L(v-u) \tag{29}$$

From Eqs. (8) and (9) we have

$$\int_\Omega \sum_{k=1}^{2}\left\{\sum_{j=1}^{2}\left[\left(\frac{\partial \tilde{f}_k}{\partial P_c}\right)^{\mathrm{T}} \tilde{D}\left(\frac{\partial \tilde{g}_j}{\partial P_c}\right)\tilde{\lambda}_j - \frac{\partial \tilde{f}_k}{\partial P_c}\tilde{D}\tilde{\varepsilon}(u) - \tilde{f}_k^0\right]\right\}(\tilde{r}_k - \tilde{\lambda}_k)\,\mathrm{d}\Omega \geqslant 0 \tag{30}$$

or

$$\tilde{c}(\tilde{\lambda}, \tilde{r} - \tilde{\lambda}) - \tilde{d}(u, \tilde{r} - \tilde{\lambda}) + \tilde{j}(\tilde{r} - \tilde{\lambda}) \geqslant 0, \quad \forall \tilde{r} \in \tilde{L}_2(\Omega), \;\; \tilde{r}_k \geqslant 0 \tag{31}$$

Similarly, from Eqs. (17) and (18) we can write

$$\int_\Omega \sum_{\alpha=1}^{m}\left\{\sum_{\beta=1}^{m}\left[\left(\frac{\partial f_\alpha}{\partial \sigma}\right) D\left(\frac{\partial g_\beta}{\partial \sigma}\right) + t_{\alpha\beta}\right]\lambda_\beta - \left(\frac{\partial f_\alpha}{\partial \sigma}\right) D\varepsilon(u) - f_\alpha^0\right\}(r_\alpha - \lambda_\alpha)\,\mathrm{d}\Omega \geqslant 0$$
$$\forall r \in \overline{L}_2(\Omega), \;\; r_\alpha \geqslant 0, \;\; \alpha = 1, 2, \ldots, m \tag{32}$$

or

$$c(\lambda, r-\lambda) - d(u, r-\lambda) + j(r-\lambda) \geqslant 0 \quad \forall r \in \overline{L}_2(\Omega), \;\; r_\alpha \geqslant 0 \tag{33}$$

Then the inequality formulation (23) can be obtained by summing up Eqs. (29), (31) and (33).

Otherwise, taking $\{v, r, \tilde{r}\} = \{u \pm \omega, \lambda, \tilde{\lambda}\} \in \tilde{K}$ in Eq. (23), and considering the bilinearity of $a(u, v)$, $\tilde{a}(u, v)$, $b(v, r)$, $\tilde{b}(u, \tilde{\lambda})$ and $L(v)$, it gives

$$a(u, \omega) - b(\omega, \lambda) + \tilde{a}(u, \omega) - \tilde{b}(\omega, \tilde{\lambda}) = L(\omega) \tag{34}$$

which is identical with Eq. (29). And thus the satisfaction of Eqs. (19)–(22) is proved easily.

Eqs. (34) and (23) yield

$$c(\lambda, r-\lambda) - d(u, r-\lambda) + j(r-\lambda) + \tilde{c}(\tilde{\lambda}, \tilde{r}-\tilde{\lambda}) - \tilde{d}(u, \tilde{r}-\tilde{\lambda}) + \tilde{j}(\tilde{r}-\tilde{\lambda}) \geqslant 0 \tag{35}$$

Setting $r = \lambda$, we have

$$\tilde{c}(\tilde{\lambda}, \tilde{r}-\tilde{\lambda}) - \tilde{d}(u, \tilde{r}-\tilde{\lambda}) + \tilde{j}(\tilde{r}-\tilde{\lambda}) \geqslant 0 \tag{36}$$

Setting $\tilde{r} = 2\tilde{\lambda}$, $\tilde{r} = 0$ (and consequently $\tilde{r} \in \tilde{L}_2(\Omega)$) we get

$$\tilde{c}(\tilde{\lambda}, \tilde{\lambda}) - \tilde{d}(u, \tilde{\lambda}) + \tilde{j}(\tilde{\lambda}) = 0 \tag{37}$$

which leads to Eq. (9) by expansion.

Substituting Eq. (37) into Eq. (36), we have

$$\tilde{c}(\tilde{\lambda}, \tilde{r}) - \tilde{d}(u, \tilde{r}) + \tilde{j}(\tilde{r}) \geqslant 0 \tag{38}$$

which leads to Eq. (8) by expansion.

In the similar way, setting $\tilde{r} = \tilde{\lambda}$, $r = 2\lambda$, $r = 0$ in Eq. (35), we obtain

$$c(\lambda, r-\lambda) - d(u, r-\lambda) + j(r-\lambda) \geqslant 0 \tag{39}$$

$$c(\lambda,\lambda) - d(u,\lambda) + j(\lambda) = 0 \tag{40}$$

Consequently, it gives

$$c(\lambda, r) - d(u, r) + j(r) \geqslant 0 \tag{41}$$

Making expansion of Eqs. (41) and (40), we can get Eqs. (17) and (18). Now the equivalence of these two forms is proved.

4. The finite element discretization and the solution by quadratic programming

4.1. Finite element discretization

The body Ω under investigation is divided into E_s elements, including n_1 ($n_1 \leqslant E_s$) elastoplastic elements. The yield postulate of the eth element consists of L_e ($L_e \geqslant 1$) smooth yield conditions, and then the number of the state equations for the system is $L = \sum_{e=1}^{n_1} L_e$, the contact boundary Γ_c is divided into n_2 contact elements $\Gamma_c = \sum_{e=1}^{n_2} \Gamma_c^e$. The number of sliding conditions for the eth element is l_e, and is $l = \sum_{e=1}^{n_2} l_e$ totally.

Introducing the operator B, the shape functions N for the displacement and $\tilde{N}$ for the relative displacement, we obtain the inequality form (23) in finite element discretization as

$$(\varphi - \delta)^{\mathrm{T}}(K\delta - \Phi\lambda - q) - (r - \lambda)^{\mathrm{T}}(C\delta - U\lambda - d) \geqslant 0 \tag{42}$$

where

$$K = \sum_{e=1}^{E_s} \int_{\Omega^e} B^{\mathrm{T}} D B \,\mathrm{d}\Omega + \sum_{e=1}^{n_2} \int_{\Gamma_c^e} \tilde{N}^{\mathrm{T}} \tilde{D} \tilde{N} \,\mathrm{d}\Gamma$$

$$q = \sum_{e=1}^{E_s} \left\{ \int_{\Omega^e} N^{\mathrm{T}} f \,\mathrm{d}\Omega + \int_{\Gamma_P^e} N^{\mathrm{T}} P \,\mathrm{d}\Gamma \right\}$$

$$\Phi = \sum_{e=1}^{n_1} \sum_{\alpha=1}^{L_e} \int_{\Omega^e} \left[\left(\frac{\partial g^e}{\partial \sigma} \right)^{\mathrm{T}} D B \right]^{\mathrm{T}} \mathrm{d}\Omega + \sum_{e=1}^{n_2} \sum_{\alpha=1}^{l_e} \int_{\Gamma_c^e} \left[\left(\frac{\partial \tilde{g}^e}{\partial P_c} \right)^{\mathrm{T}} \tilde{D} \tilde{N} \right]^{\mathrm{T}} \mathrm{d}\Gamma$$

$$C = \sum_{e=1}^{n_1} \sum_{\alpha=1}^{L_e} \int_{\Omega^e} \left(\frac{\partial f^e}{\partial \sigma} \right)^{\mathrm{T}} D B \,\mathrm{d}\Omega + \sum_{e=1}^{n_2} \sum_{\alpha=1}^{l_e} \int_{\Gamma_c^e} \left(\frac{\partial \tilde{f}^e}{\partial P_c} \right)^{\mathrm{T}} \tilde{D} \tilde{N} \,\mathrm{d}\Gamma$$

$$U = \sum_{e=1}^{n_1} \sum_{\alpha=1}^{L_e} \int_{\Omega^e} \left[\left(\frac{\partial f^e}{\partial \sigma} \right) D \left(\frac{\partial g^e}{\partial \sigma} \right) + t \right] \mathrm{d}\Omega + \sum_{e=1}^{n_2} \sum_{\alpha=1}^{l_e} \int_{\Gamma_c^e} \left[\left(\frac{\partial \tilde{f}^e}{\partial P_c} \right) \tilde{D} \left(\frac{\partial \tilde{g}^e}{\partial P_c} \right) \right] \mathrm{d}\Gamma$$

$$d = -\sum_{e=1}^{n_1} \sum_{\alpha=1}^{L_e} \int_{\Omega^e} f_\alpha^{0e} \,\mathrm{d}\Omega - \sum_{e=1}^{n_2} \sum_{\alpha=1}^{l_e} \int_{\Gamma_c^e} \tilde{f}_\alpha^{0e} \,\mathrm{d}\Gamma$$

$$\lambda = \left\{ \lambda_\alpha^e \middle| \begin{array}{ll} \alpha = 1, 2, \ldots, L_e, & e \leqslant n_1 \\ \alpha = 1, 2, \ldots, l_e, & e > n_1 \end{array} \quad e = 1, 2, \ldots, n_1 + n_2 \right\}$$

Taking notice of the arbitrariness of $\{\varphi, r\}$, the following form, equivalent to Eq. (42), can be derived as

$$K\delta - \Phi\lambda - q = 0 \tag{43a}$$

$$C\delta - U\lambda - d \leqslant 0 \tag{43b}$$

$$\lambda^{\mathrm{T}}(C\delta - U\lambda - d) = 0 \tag{43c}$$

Involving a relaxation variable $v \geqslant 0$, Eqs. (43b) and (43c) will be

$$C\delta - U\lambda - d + v = 0 \tag{44a}$$

$$\lambda^{\mathrm{T}}v = \lambda^{\mathrm{T}}(-C\delta + U\lambda + d) = 0 \tag{44b}$$

Then Eqs. (43a), (44a) and (44b) can be written as

$$\begin{bmatrix} v \\ 0 \end{bmatrix} + \begin{bmatrix} -U & C \\ -\Phi & K \end{bmatrix} \begin{bmatrix} \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} d \\ q \end{bmatrix} \tag{45}$$

$$v^{\mathrm{T}}\lambda = 0, \quad v \geqslant 0, \quad \lambda \geqslant 0 \tag{46}$$

The above equations are the finite element discrete formulae of the elastoplastic contact problem. In fact, it can also be described in a form of the linear complementary problem with a free variable, as it will be seen in the next section.

4.2. The algorithm of quadratic programming

It is known that a standard quadratic programming problem is defined as (see e.g. Heesterman (1983)):

$$\text{minimize } f(x) = \frac{1}{2}x^{\mathrm{T}}Ax + b^{\mathrm{T}}x \tag{47}$$

$$\text{subject to}: Cx \leqslant d \tag{48}$$

$$x \geqslant 0 \tag{49}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $C \in \mathbb{R}^{m\times n}$, $d \in \mathbb{R}^m$.

Based on Kuhn–Tucker's necessary and sufficient conditions of optimum solution, the standard quadratic programming is equivalent to the following complementary problem.

The complementary problem is defined as:

Find $w \in \mathbb{R}^n$, $z \in \mathbb{R}^n$, satisfying

$$\begin{aligned} & w - Mz = q \\ & w^{\mathrm{T}}z = 0, \quad w \geqslant 0, \quad z \geqslant 0 \end{aligned} \tag{50}$$

where $M \in \mathbb{R}^{n\times n}$, $q \in \mathbb{R}^n$ are given values.

Eq. (50) can be solved by the Lemke algorithm (see e.g. Ravindran and Lee (1981)).

In actual problems there are no special requirements of non-negativity for the variables like reaction force, displacement, stress, etc. So the quadratic programming problem is described as:

Assuming $x \in \mathbb{R}^n$

$$\text{minimize } f(x) = \frac{1}{2}x^{\mathrm{T}}Ax + b^{\mathrm{T}}x \tag{51}$$

$$\text{subject to : } Cx \leqslant d \tag{52}$$

where $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ are given values.

Comparing with Eqs. (47)–(49), we have no condition of $x \geqslant 0$ here, so we may write Eqs. (51) and (52) in form of a standard quadratic programming problem by using the transformation of

$$x = x^{(1)} - x^{(2)}, \quad x^{(1)} \geqslant 0, \quad x^{(2)} \geqslant 0.$$

However, the number of variables will be two times more, resulting in more requirements of CPU time and storage. Thus a new approach is worth to be recommended.

Assuming that A is a positive definite matrix and C is a full-rank matrix and involving the Lagrange multiplier $\lambda \in \mathbb{R}^m$ to relax the constraint conditions (52), we obtain

$$L = \frac{1}{2}x^{\mathrm{T}}Ax + b^{\mathrm{T}}x + \lambda^{\mathrm{T}}(Cx - d) \tag{53}$$

According to Kuhn–Tucker's conditions we have $\lambda \geqslant 0$ and may derive the following relations:

$$Ax + b + C^{\mathrm{T}}\lambda = 0 \tag{54}$$

$$Cx - d \leqslant 0 \tag{55}$$

$$\lambda^{\mathrm{T}}(Cx - d) = 0 \tag{56}$$

Introducing a relaxation variable $v \geqslant 0$ ($v \in \mathbb{R}^m$). Eqs. (54)–(56) may be rewritten as

$$\begin{cases} \begin{bmatrix} v \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & -C \\ C^{\mathrm{T}} & A \end{bmatrix} \begin{bmatrix} \lambda \\ x \end{bmatrix} = \begin{bmatrix} d \\ b \end{bmatrix} \\ \lambda^{\mathrm{T}}v = 0, \quad \lambda \geqslant 0, \quad v \geqslant 0, \quad x \in \mathbb{R}^n \end{cases} \tag{57}$$

Comparing with Eq. (50), here the condition $x \geqslant 0$ is in default.

Eq. (57) describe the linear complementary problem with a free variable. Compared with Eqs. (45) and (46), these two sets of equations have no difference in form. It can be solved by a new approach of quadratic programming.

Firstly, transforming the augmented matrix of Eq. (57)

$$\begin{bmatrix} I_m & 0 & C & d \\ 0 & -C^{\mathrm{T}} & -A & b \end{bmatrix} \tag{58}$$

and making the left multiplication of matrix $\begin{bmatrix} I_m & CA^{-1} \\ 0 & A^{-1} \end{bmatrix}$ we have

$$\begin{bmatrix} I_m & -CA^{-1}C^{\mathrm{T}} & 0 & d + CA^{-1}b \\ 0 & -A^{-1}C^{\mathrm{T}} & I_n & A^{-1}b \end{bmatrix} \tag{59}$$

Setting

$$P = A^{-1}C^{\mathrm{T}}, \quad b' = A^{-1}b$$
$$Q = CP, \quad d' = d + Cb$$

Eq. (57) can be turned to a new linear complementary problem:

$$\begin{cases} \begin{bmatrix} v \\ 0 \end{bmatrix} - \begin{bmatrix} Q & 0 \\ P & I_n \end{bmatrix} \begin{bmatrix} \lambda \\ x \end{bmatrix} = \begin{bmatrix} d' \\ b' \end{bmatrix} \\ \lambda^{\mathrm{T}} v = 0, \quad \lambda \geqslant 0, \quad v \geqslant 0 \end{cases} \tag{60}$$

Secondly, from Eq. (60), we can get an initial solution as:

$$v = d', \quad \lambda = 0, \quad x = -b'$$

If $d' \geqslant 0$ all conditions are satisfied, that means the solution is found simply.

If $d' < 0$ the solution should be found for a new linear complementary system by introducing an artificial variable Z_0 ($Z_0 = -\min\{d'_i, i = 1, 2, \ldots, m\}$).

$$\begin{cases} \begin{bmatrix} v \\ 0 \end{bmatrix} - \begin{bmatrix} Q & 0 \\ P & I_n \end{bmatrix} \begin{bmatrix} \lambda \\ x \end{bmatrix} - \begin{bmatrix} Z_0 i_m \\ 0 \end{bmatrix} = \begin{bmatrix} d' \\ b' \end{bmatrix} = q \\ \lambda^{\mathrm{T}} v = 0, \quad \lambda \geqslant 0, \quad v \geqslant 0, \quad Z_0 \geqslant 0 \end{cases} \tag{61}$$

Thus we may obtain an almost complementary basic feasible solution:

$$v = d' + Z_0 i_m, \quad \lambda = 0, \quad x = -b' \tag{62}$$

Eq. (62) is not a feasible solution of the original problem due to the involvement of Z_0. The solution of Eq. (57) can then be found by use of Lemke algorithm, in which the base exchange between λ and v should be made until Z_0 is being removed.

In solution by the quadratic programming the main workload is the exchange between the complementary bases λ and v. A base exchanging operation is equivalent to that for a Gauss Jordan matrix elimination. Even if all the elements enter the plastic state, only S ($S \leqslant L + 1$) operations are needed. Similarly, for the contact state equations only S' ($S' \leqslant l + 1$) operations are used. Hence, totally there are $S + S'$ ($\leqslant L + l + 2$) base exchanges at most. It shows great advantages in convergence and convenience, compared with the conventional iterative methods, where both the yield conditions and the contact relations should be diagnosed in every iteration.

5. Numerical examples

To show the applicability and effectiveness of the theory described in this paper, three test examples are presented.

Example (1) – Contact problem between two thick tubes with infinite length

The classical contact problem of two thick tubes with infinite length is studied. The tubes are pushed together by the forces P as shown in Fig. 2. Young's modulus E, Poisson's ratio v, the inner radius r and the outer radius R were taken equal to 9.8×10^8 N/m^2, 0.3, 10.0 and 16.0 cm.

The contact zone is what we should follow with interest. The discretization is made for the region shown in Fig. 3.

Fig. 4 shows the finite element mesh of the contact region (shadowed in Fig. 3).

The results for the contact radius are given in Table 1. Comparing with the Hertz solution, the relative error is caused mainly by the discretization of the contact boundary.

The comparison for the contact force is shown in Fig. 5. It must be pointed out that, when P equals 1.96×10^3 N/m, the numerical result obtained by the author is a little bit smaller than that of the Hertz's. But in this case the Hertz's solution is not an exact one when a few elements have already entered the plastic stage.

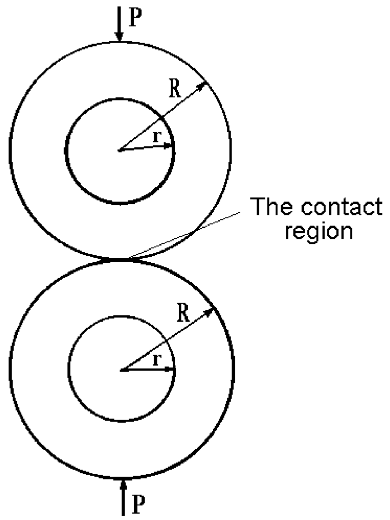

Fig. 2. Two thick tubes with infinite length.

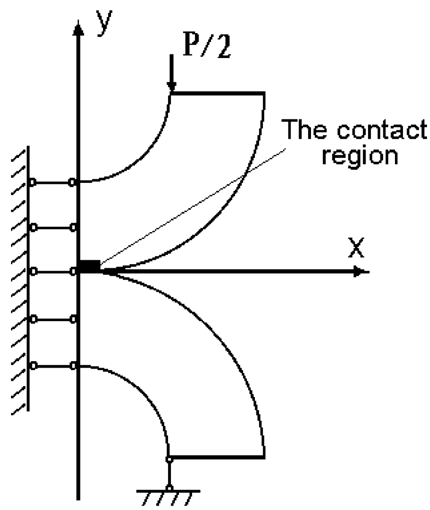

Fig. 3. The location of the contact region.

Example (2) – The contact problem of a hexahedron rock specimen subjected to pressure

A hexahedron rock specimen, subjected to pressure by two rigid plates (see Fig. 6) is studied. Assume that the coefficients of friction $\overline{\mu}$ between the specimen and the rigid plates are the same, the initial interval $a_0 = 0$, the material of the specimen follows the Mohr–Coulomb yield postulate

$$f = \tau + \sigma_n \tan\varphi - C \leqslant 0$$
$$g = \tau + \psi\sigma_n \tan\varphi + t$$

where ψ is a factor of expansion.

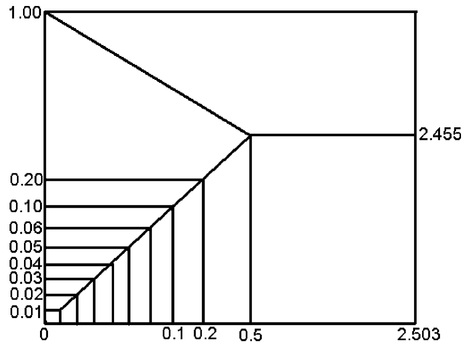

Fig. 4. The finite element mesh of the contact region.

Table 1
The contact radius (cm)

P (N/m)	Contact radius		
	Hertz solution	Present method	Relative error
0.98×10^3	0.0434	0.04	7.10%
1.96×10^3	0.061	0.06	1.67%

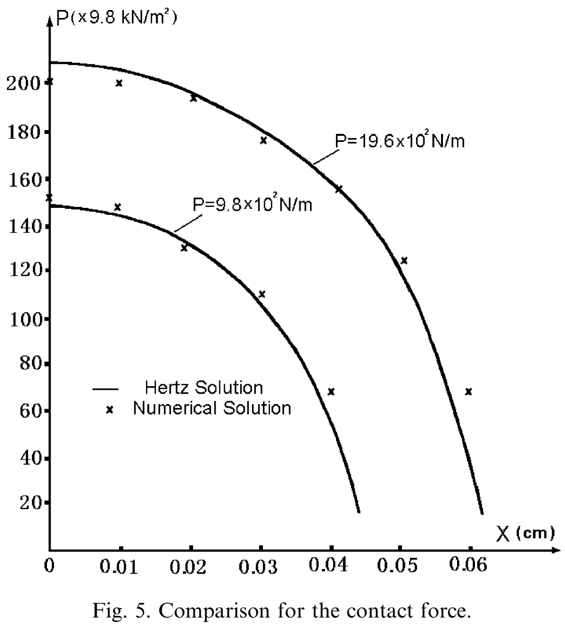

Fig. 5. Comparison for the contact force.

When $\psi = 1$, this form yields the associated postulate, and for $\psi = 0$, the non-associated postulate without plastic expansion.

In the example, $\varphi = 40°$, $C = 0.2$, $E = 5 \times 10^3$ Pa, $\mu = 0.25$. The influence of $\overline{\mu}$ on the elastoplastic state at $\psi = 1$ and 0 is analyzed.

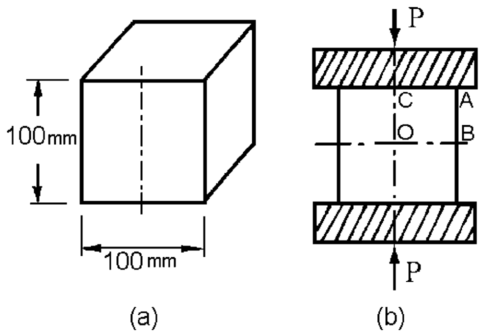

Fig. 6. Regular hexahedron rock specimen subjected to the pressure.

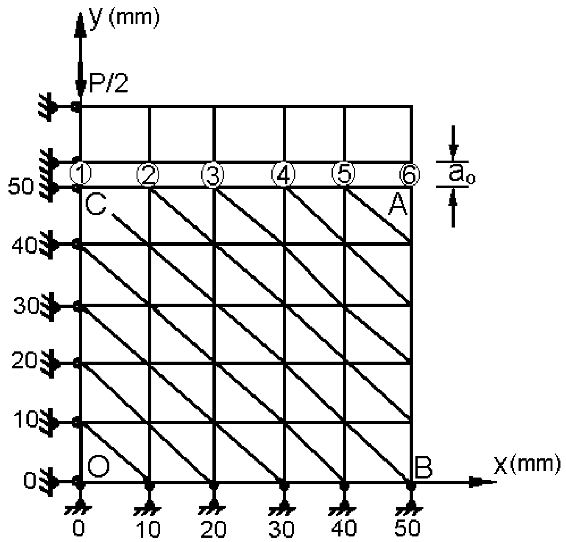

Fig. 7. Discretization of the specimen.

The discretization is made for 1/4 region because of the symmetry. The studied region of the specimen is divided into 50 triangular elements, five rigid elements for the pressure plate and six contact elements between them (see Fig. 7).

Fig. 8 depicts the horizontal and vertical deformations of the boundary AB. It is seen from the figure that the deformations are greatly influenced by the coefficient $\overline{\mu}$. Under the same pressure P the horizontal displacement of point A for $\overline{\mu} = 0.1$ is of 25 times greater than that for $\overline{\mu} = 0.2$, $\psi = 0$ and is of 36 times greater than that for $\overline{\mu} = 0.2$, $\psi = 1$. In case when $\psi = 0$ the horizontal deformation for the material following the non-associated postulate is smaller than that following the associated postulate of $\psi = 1$, but, on contrary, the vertical deformation is greater.

Fig. 9 gives the distribution of plastic region ($\overline{\mu} = 0.1$, 0.2, 0.4, 0.5) under $P = 124$ N. The result was calculated in five increment steps. Obviously, the smaller $\overline{\mu}$ is, the more damage possibility for the specimen. Compared with $\overline{\mu}$, the effect of ψ is much smaller. Nevertheless, the influence of $\overline{\mu}$ reduces rapidly when $\overline{\mu} \geqslant 0.3$, and can be ignored for $\overline{\mu} > 0.5$.

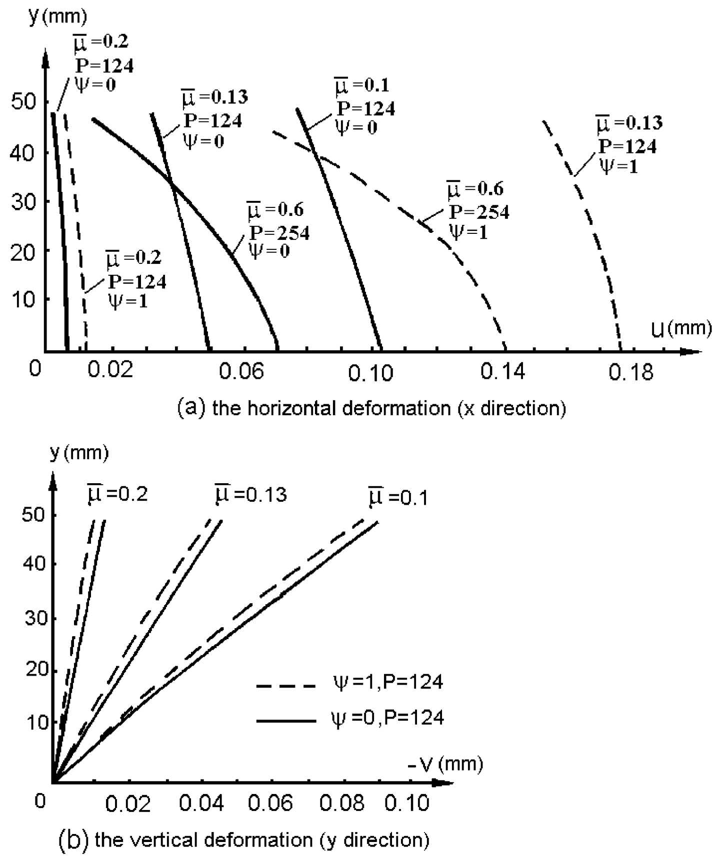

Fig. 8. The deformations on the boundary AB.

When $\overline{\mu} \leqslant 0.4$, the two contact bodies slide in the region of the contact elements 3, 4, 5, 6. The slide region decreases to the zone of elements 5, 6 when $\overline{\mu} = 0.5$. And only one element remains when $\overline{\mu} = 0.6$. The sliding will end with the further increasing of $\overline{\mu}$. It is worth noticing that the value of P shows no influence on sliding.

Table 2 shows the numerical values of the contact force P_n. Though P_n for elements has different values with different $\overline{\mu}$, the total force keeps constant. ($P_\tau = \overline{\mu} P_n$, omitted in Table 2).

Applying the new approach presented in this paper, fewer base exchanges are required in every increment, and the result shows great advantages in computational effectiveness and more convenient than other iterative methods.

Example (3) – The load-bearing capacity of semi-infinite foundation

The load-bearing capacity is studied for a semi-infinite foundation, subjected to the pressure of a ridged strip on it without friction (Fig. 10). The discretization is made for 1/2 region.

Fig. 11 gives the P–δ relation at the center line of the load P, where curve $OABE$ is obtained under the associated postulate ($\psi = 1$) and OCD under the non-associated postulate without plastic expansion ($\psi = 0$).

In Fig. 11, it is seen that the curves OAB and OCD are in good accordance with that of the classical solution. The result from O to E was computed by 13 load increments and from O to D by 11 increments.

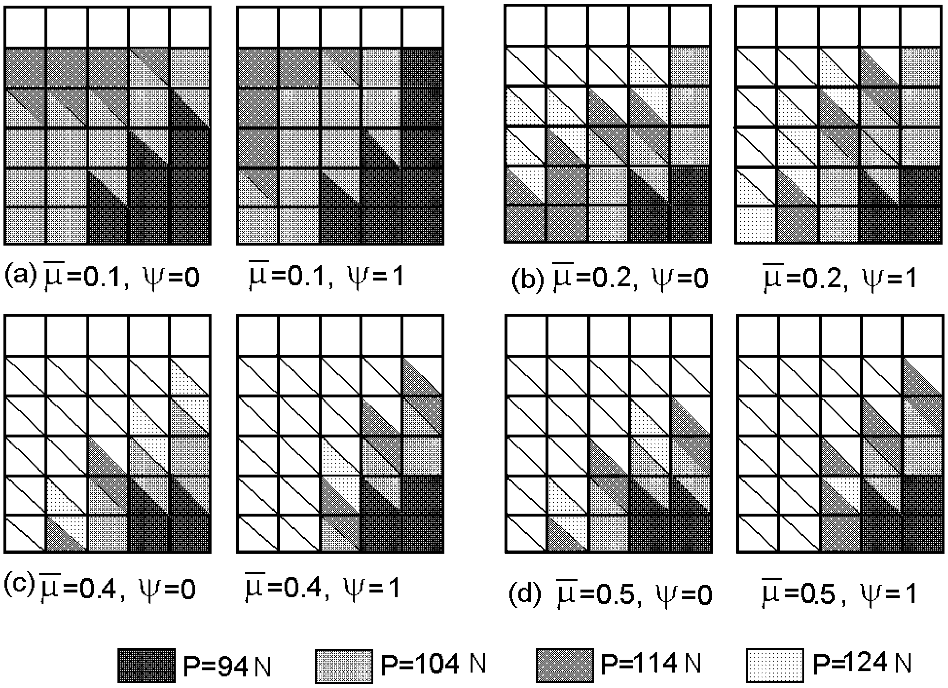

Fig. 9. The distribution of plastic region.

Table 2
Numerical values of the contact force P_n

$\overline{\mu}$	ψ	Element number						Total forces
		1	2	3	4	5	6	
0.1	0.0	−5.0	−12.8	−12.4	−12.7	−13.0	−6.2	−62.1
	1.0	−5.2	−12.7	−12.0	−12.6	−12.8	−6.6	−61.9
0.2	0.0	−6.4	−12.6	−11.9	−11.6	−12.3	−7.1	−61.9
	1.0	−6.5	−12.7	−11.9	−11.6	−12.0	−7.3	−62.0
0.4	0.0	−6.0	−12.3	−11.9	−11.0	−11.1	−9.6	−61.9
	1.0	−5.8	−12.0	−11.8	−11.0	−10.9	−10.4	−61.9
0.5	0.0	−6.0	−12.2	−11.9	−11.0	−11.2	−9.6	−61.9
	1.0	−5.8	−12.0	−11.8	−11.0	−10.9	−10.4	−61.9

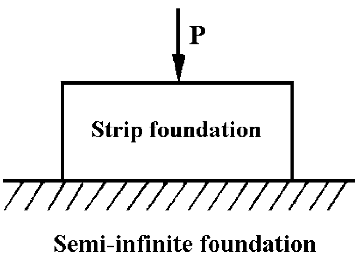

Fig. 10. Semi-infinite foundation subjected to pressure.

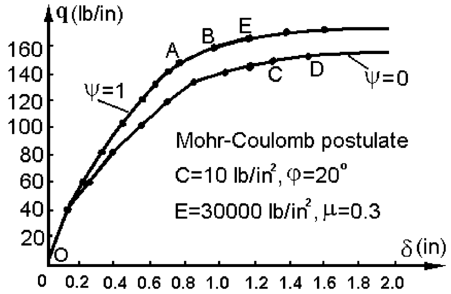

Fig. 11. P–δ relation.

Let us compare with the numerical solutions obtained by Chen (1982) and Zienkiewicz et al. (1975): Chen (1982) used 44 increment steps for the computation from O to A, 269 steps from A to B, yet the iteration is required in every step of computation to avoid divergence, and, finally, the evaluation fully diverged at B and D. And, for the same example, Zienkiewicz (1975) showed his evaluation fully diverged at E and D. On the contrary, using the presented approach of the quadratic programming, the solution does not diverge along with loading even when the deformation increases rapidly enough. It is obvious that the presented scheme has great advantages in convergence and convenience.

6. Conclusions

1. In this paper, the unified description for the contact problem and the elastoplastic problem is made by the variational inequality model. A corresponding functional with the relaxed constitutive relation and the contact restriction is formed. The new forms have reliable mathematical basis.
2. The quadratic programming is adopted to transform the non-linear problem to a complementary linear problem for the sake of simplicity in computation.
3. The numerical results show that the present approach has several advantages in accuracy, convergence and computational effectiveness, and it is more convenient than other iterative methods and other traditional methods.
4. The method presented in this paper can be used for other non-linear problems with the constitutive equations in forms of inequalities.

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